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Fritz John Second-Order Duality for Nonlinear Programming

I. HUSAIN
Department of Mathematics, Regional Engineer College
Srinagar 190006, Kashmir, India

N. G. RUEDA
Department of Mathematics, Merrimack College
North Andover, MA 01845, U.S.A.

Z. JABEEN
Department of Mathematics, Regional Engineer College
Srinagar 190006, Kashmir, India

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Abstract—A second-order dual to a nonlinear programming problem is formulated. This dual uses the Fritz John necessary optimality conditions instead of the Karush-Kuhn-Tucker necessary optimality conditions, and thus, does not require a constraint qualification. Weak, strong, strict-converse, and converse duality theorems between primal and dual problems are established.

Keywords—Fritz John second-order duality, Bonvex, Pseudobonvex, Strictly pseudobonvex, Semi-strictly pseudobonvex.

1. INTRODUCTION

Consider the nonlinear programming problem

$$\begin{aligned} &\text{minimize} && f(x), && \text{(NP)} \\ &\text{subject to} && g(x) \leq 0, && (1) \end{aligned}$$

where $x \in R^n$, f and g are twice differentiable functions from R^n into R and R^m, respectively. Mangasarian [1] formulated the following Wolfe type [2] second-order dual of (NP):

$$\begin{aligned} &\text{maximize} && \left[f(u) + y^\top g(u)\right] - \frac{1}{2} p^\top \nabla^2 \left[f(u) + y^\top g(u)\right] p, && (\text{ND}_1) \\ &\text{subject to} && \nabla \left[f(u) + y^\top g(u)\right] + \nabla^2 \left[f(u) + y^\top g(u)\right] p = 0, \qquad y \geq 0, \end{aligned}$$

where $p \in R^n$ and for any function $\phi : R^n \to R$, the symbol $\nabla^2 \phi(x)$ designates $n \times n$ symmetric matrix of second-order partial derivatives. Mangasarian [1] established usual duality theorems between (NP) and (ND_1) under assumptions that are involved and rather difficult to verify.

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In [3], Mond established the duality of (ND_1) to (NP) under simpler assumptions, i.e., if for all (x, u, p),

$$f(x) - f(u) \geq (x-u)^\top \left[\nabla f(u) + \nabla^2 f(u) p\right] - \frac{1}{2} p^\top \nabla^2 f(u) p, \tag{2}$$

$$g_i(u) - g_i(u) \geq (x-u)^\top \left[\nabla g_i(u) + \nabla^2 g_i(u) p\right] - \frac{1}{2} p^\top \nabla^2 g_i(u) p, \qquad i = 1, 2, \dots, m. \tag{3}$$

Later, Mond and Weir [4] extended definitions (2) and (3), respectively, to

$$(x-u)^\top \left[\nabla f(u) + \nabla^2 f(u) p\right] \geq 0 \Rightarrow f(x) \geq f(u) - \frac{1}{2} p^\top \nabla^2 f(u) p, \tag{4}$$

$$g_j(x) \geq g_j(u) - \frac{1}{2} p^\top \nabla^2 g_j(u) p \Rightarrow (x-u)^\top \left[\nabla^2 g_j(u) + \nabla^2 g_j(u) p\right] \geq 0, \quad j = 1, 2, \dots, m. \tag{5}$$

Calling (4) and (5) second-order pseudoconvexity and second-order quasiconvexity, respectively, Mond and Weir [4] established duality of the following dual (ND_2) to (NP) Bector and Chandra [5] renamed second-order convex, second-order pseudoconvex, and second-order quasiconvex, respectively, as bonvex, pseudobonvex, and quasibonvex:

$$\begin{aligned} \text{Maximize} \quad & f(u) - \frac{1}{2} p^\top \nabla^2 f(u) p, && (\mathrm{ND}_2) \\ \text{Subject to} \quad & \nabla \left[f(u) + y^\top g(u)\right] + \nabla^2 f(u) + y^\top g(u)]p = 0, \\ & y^\top g(u) - \frac{1}{2} p^\top \nabla^2 \left(y^\top g(u)\right) p \geq 0, \qquad y \geq 0. \end{aligned}$$

Mond [3] established strong duality between (NP) and (ND_1) using Karush-Kuln-Tucker [2] optimality conditions. Mond and Weir [4] proved strong and strict-converse duality theorems between (NP) and (ND_2) that also utilized the Karush-Kuhn-Tucker [2] condition. So the duality results validated both in [3] and [4] require that the constraints must satisfy some sort of constraint qualification.

In this paper, we formulate a dual problem to the problem (NP) using Fritz John [2] necessary optimality conditions instead of Karush-Kuhn-Tucker [2] optimality conditions, and establish weak, strong, Mangasarian [2] type strict-converse and Huard [2] converse duality theorems. Thus, the requirement of a constraint qualification is eliminated.

2. SECOND-ORDER DUALITY

We present a different dual to (NP) and establish strong and strict converse duality using Fritz John [2] optimality conditions at the optimal point for the primal. Thus, the need for a constraint qualification is eliminated.

Consider the problem

$$\begin{aligned} \text{Maximize} \quad & f(u) - \frac{1}{2} p^\top \nabla^2 f(u) p, && (\mathrm{ND}) \\ \text{Subject to} \quad & r\left(\nabla f(u) + \nabla^2 f(u) p\right) + \nabla (y^\top g(u) + \nabla^2 \left(y^\top g(u)\right) p = 0, && (5) \\ & y^\top g(u) - \frac{1}{2} p^\top \nabla^2 \left(y^\top g(u)\right) p \geq 0, && (6) \\ & (r, y) \geq 0, && (7) \\ & (r, y) \neq 0. && (8) \end{aligned}$$

Before proceeding to the main results of this section, we give the following definitions.

DEFINITION 2.1. *A twice differentiable function f will be said to be strictly pseudobonvex at x^*, if for all $x \neq x^*$,*

$$(x - x^*)^\top \left[\nabla f(x^*) + \nabla^2 f(x^*)p\right] \geq 0 \Rightarrow f(x) > f(x^*) - \frac{1}{2}p^\top \nabla^2 f(x^*)p.$$

For $y \in R^m$ and a twice differentiable vector function g, $R^n \to R^m$, $y^\top g(.)$ is said to be semi-strictly pseudobonvex, if $y^\top g(.)$ is strictly pseudobonvex for $y \geq 0$ and $y \neq 0$.

If $p = 0$, the definition of semi-strict pseudobonvexity becomes that of semi-strict pseudoconvexity incorporated by Weir and Mond [6].

THEOREM 2.1. WEAK DUALITY. *Let x satisfy the constraints of (NP) and (r, u, y, p) satisfy the constraints of (ND). If $f(.)$ is pseudobonvex and $y^\top g(.)$ is semi-strictly pseudobonvex for all feasible (r, x, u, y, p), then*

$$\text{infimum (NP)} \geq \text{supremum (ND)}.$$

PROOF. Suppose that

$$f(x) < f(u) - \frac{1}{2}p^\top \nabla^2 f(u)p.$$

This, in view of pseudobonvexity of $f(.)$, yields

$$(x - u)^\top \left[\nabla f(u) + \nabla^2 f(u)p\right] < 0.$$

Thus,

$$(x - u)^\top r \left[\nabla f(u) + \nabla^2 f(u)p\right] \leq 0, \tag{9}$$

with strict inequality in (9) if $r > 0$.

From the constraints of (NP) and (ND),

$$y^\top g(x) \leq y^\top g(u) - \frac{1}{2}p^\top \nabla^2 \left(y^\top g(u)\right) p.$$

This, because of semi-strict pseudobonvexity of $y^\top g(.)$, gives

$$(x - u)^\top \left[\nabla \left(y^\top g(u)\right) + \nabla^2 \left(y^\top g(u)\right) p\right] \leq 0, \tag{10}$$

with strict inequality in (10), if some $y_j > 0$, $j \in \{1, 2, \dots, m\}$. Combining (9) and (10), we have

$$(x - u)^\top \left[r \left(\nabla f(u) + \nabla^2 f(u)p\right) + \nabla \left(y^\top g(u)\right) + \nabla^2 \left(y^\top g(u)\right) p\right] < 0,$$

which contradicts (5). Hence,

$$f(x) \geq f(u) - \frac{1}{2}p^\top \nabla^2 f(u)p.$$

That is,

$$\text{infimum (NP)} \geq \text{supremum (ND)}.$$

THEOREM 2.2. STRONG DUALITY. *If x^* is an optimal solution for (NP), then there exist $r \in R$ and $y \in R^m$ such that $(r, x^*, y, p = 0)$ is feasible for (ND), and the corresponding values of (NP) and (ND) are equal. If, for all feasible (r, x, u, y, p), $f(.)$ is pseudobonvex and $y^\top g(.)$ is semi-strictly pseudobonvex, then (r, x^*, y, p) is an optimal solution of (ND).*

PROOF. Since x^* is an optimal solution of (NP), by the Fritz John necessary optimality conditions [2], there exist $r \in R$ and $y \in R^m$ such that

$$r^* \nabla f(x^*) + \nabla \left({y^*}^\top g(x^*)\right) = 0,$$

$${y^*}^\top g(x^*) = 0, \qquad (r^*, y^*) \geq 0, \qquad (r^*, y^*) \neq 0.$$

So $(r^*, x^*, y^*, p^* = 0)$ is feasible for (ND), and the corresponding values of (NP) and (ND) are equal. If $f(.)$ is pseudobonvex and $y^\top g(.)$ is semi-strictly pseudobonvex, then by Theorem 2.1, $(r^*, x^*, y^*, p^* = 0)$ is an optimal solution of (ND).

We now first give a Mangasarian type [2] strict converse duality theorem (Theorem 2.2) for the dual (ND) to (NP).

THEOREM 2.3. STRICT CONVERSE DUALITY. *Let $f(.)$ be strictly pseudobonvex and $y^\top g(.)$ semi-strictly pseudobonvex. Let x^* be an optimal solution of (NP). If (r^*, x^*, y^*, p^*) is an optimal solution of (ND), then $x^* = u^*$, i.e., u^* is an optimal solution of (NP).*

PROOF. We assume that $u^* \neq x^*$ and exhibit a contradiction. Since x^* is an optimal solution of (NP), it follows by strong duality (Theorem 2) that there exist $y^* \in R^m$, $p^* = 0$ such that $(x^*, y^*, p^* = 0)$ is an optimal solution of (ND). Since (r^*, u^*, y^*, p^*) is also an optimal solution of (ND), it follows that

$$f(x^*) = f(u^*) - \frac{1}{2} p^{*\top} \nabla^2 f(u^*) p^*.$$

This, in view of strict pseudobonvexity of $f(.)$, gives

$$(x^* - u^*)^\top \left[\nabla f(u^*) + \nabla^2 f(u^*) p^*\right] < 0.$$

Also, from the constraints of (NP) and (ND), we have

$$y^{*\top} g(x^*) \leq y^{*\top} g(u^*) - \frac{1}{2} \nabla^2 \left(y^{*\top} g(u^*)\right) p^*, \tag{11}$$

which because of semi-strict pseudobonvexity yields

$$(x^* - u^*)^\top \left[\nabla \left(y^{*\top} g(u^*)\right) + \nabla^2 \left(y^{*\top} g(u^*)\right) p^*\right] \leq 0,$$

with strict inequality in (11) if $y_j^* > 0$, $j \in \{1, 2, \ldots, m\}$. From $(r^*, y^*) \geq 0$ and $(r^*, y^*) \neq 0$, we have

$$(x^* - u^*)^\top \left[r^* \left(\nabla f(u^*) + \nabla^2 f(u^*) p^*\right) + \nabla \left(y^{*\top} g(u^*)\right) + \nabla^2 \left(y^{*\top} g(u^*)\right) p^*\right] < 0,$$

contradicting the feasibility of (r^*, u^*, y^*, p^*). Hence, the result.

The following is the Huard type [2] converse duality theorem for (ND) to (NP).

THEOREM 2.4. CONVERSE DUALITY. *Let (r^*, x^*, y^*, p^*) be an optimal solution of (ND) at which*

(A1) *the $n \times n$ Hessian matrix $\nabla[r^* \nabla^2 f(x^*) + \nabla^2 (y^{*\top} g(x^*))] p^*$ is positive or negative definite,*
(A2) $\nabla(y^{*\top} g(u^*)) + \nabla^2 (y^{*\top} g(u^*)) p^* \neq 0$, *and*
(A3) *the vectors $\{[\nabla^2 f(x^*)]_j, [\nabla^2 (y^{*\top} g(x^*))]_j, j = 1, 2, \ldots, n\}$ are linearly independent, where $[\nabla^2 f(x^*)]_j$ is the j^{th} row of $[\nabla^2 f(x^*)]$ and $[\nabla^2 (y^{*\top} g(x^*))]_j$ is the j^{th} row of $[\nabla^2 (y^{*\top} g(x^*))]$.*

If, for all feasible $(r^*, x^*, u^*, y^*, p^*)$, $f(.)$ is pseudobonvex and $y^\top g(.)$ is semi-strictly pseudobonvex, then x^* is an optimal solution of (NP).

PROOF. Since (r^*, x^*, y^*, p^*) is an optimal solution of (ND), by the generalized Fritz John necessary conditions [2], there exists $\alpha \in R$, $\beta \in R^n$, $\xi \in R$, and $\eta \in R^m$ such that

$$
\begin{aligned}
&-\alpha\left\{\nabla f(x) - \frac{1}{2}p^{*\top}\left(\nabla^2 f(x^*)p^*\right)\right\} \\
&+ \beta^\top\left\{r^*\left(\nabla^2 f(x^*) + \nabla\left(\nabla^2 f(x^*)p^*\right)\right) + \nabla^2\left(y^{*\top}g(x^*)\right)\right. \\
&\left.+ \nabla(\nabla^2\left(y^{*\top}g(x^*)\right)p^*)\right\} - \theta\left\{\nabla\left(y^{*\top}g(x^*)\right) - \frac{1}{2}p^{*\top}\nabla(\nabla^2\left(y^{*\top}g(x^*)p^*\right)\right\} = 0,
\end{aligned}
\tag{12}
$$

$$\beta^\top\left(\nabla g(x^*) + \nabla^2 g(x^*)p^*\right) - \theta\left(g(x^*) - \frac{1}{2}p^{*\top}\nabla^2 g(x^*)p^*\right) - \eta = 0, \tag{13}$$

$$\beta^\top(\nabla g(x^*) + \nabla^2 g(x^*)p^* + \xi = 0, \tag{14}$$

$$(\alpha p^* + \beta r^*)^\top\left[\nabla^2 f(x^*)\right] + (\theta p^* + \beta)^\top\left[\nabla^2(y^{*\top}g(x^*)\right] = 0, \tag{15}$$

$$\eta^\top y^* = 0, \tag{17}$$

$$\xi r^* = 0, \tag{18}$$

$$(\alpha, \beta, \theta, \xi, \eta) \geq 0, \tag{19}$$

$$(\alpha, \beta, \theta, \xi, \eta) \neq 0. \tag{20}$$

Since $\{[\nabla^2 f(x^*)]_j,\ [\nabla^2(y^{*\top}g(x^*))]_j,\ j = 1, 2, \ldots, n\}$ are linearly independent at (r^*, x^*, y^*, p^*), then (15) gives

$$\alpha p^* + r^*\beta = 0 \qquad \text{and} \qquad \theta p^* + \beta = 0. \tag{21}$$

Multiplying (13) by $y^{*\top}$ and then using (16) and (17), we have

$$\beta^\top\left(\nabla\left(y^{*\top}g(x^*)\right) + \nabla^2\left(y^{*\top}g(x^*)\right)p^*\right) = 0. \tag{22}$$

Using (5) in (12), we have

$$
\begin{aligned}
&(\alpha p^* + r^*\beta)^\top\left[r^*(\nabla^2(x^*) + \nabla\left(\nabla^2 f(x^*)p\right)\right] + r^*(\theta p^* + \beta)^\top\left[\nabla^2\left(y^{*\top}g(x^*)\right)\right. \\
&\left.+ \nabla(\nabla^2\left(y^{*\top}g(x^*)\right)p^*\right] + (\alpha - r^*\theta)\left[\nabla\left(y^{*\top}g(x^*)\right) + \nabla^2\left(y^{*\top}g(x^*)\right)p^*\right] \\
&-\frac{1}{2}(\alpha p^*)r^*\nabla\left(\nabla^2 f(x^*)p^*\right) - \frac{1}{2}r^*(\theta p^*)^\top\nabla(\nabla^2\left(y^{*\top}g(x^*)p^*\right) = 0.
\end{aligned}
\tag{23}
$$

Using (21), (23) gives

$$
\begin{aligned}
&(\alpha - r^*\theta)\left(\nabla\left(y^{*\top}g(x^*)\right) + \nabla^2\left(y^{*\top}g(x^*)\right)p^*\right) \\
&+ \frac{1}{2}(\beta r^*)^\top\left\{\nabla(\nabla^2 f(x^*) - \nabla^2\left(y^{*\top}g(x^*)\right)p^*\right\} = 0.
\end{aligned}
\tag{24}
$$

Multiplying (24) by (βr^*) and using (22), we have

$$(\beta r^*)\nabla\left[\left(r^*\nabla^2 f(x^*) + \nabla^2\left(y^{*\top}g(x^*)\right)\right)p^*\right](\beta r^*) = 0.$$

By Assumption (A1) that $\nabla[r^*\nabla^2 f(x^*) + \nabla^2(y^{*\top}g(x^*)]p^*$ is positive or negative definite, it follows that

$$\beta r^* = 0.$$

In view of (A2), the equality constraint of (ND) implies $r^* \neq 0$ and so $\beta = 0$. Using $\beta = 0$ in (24), we have

$$(\alpha - r^*\theta)(\nabla(y^{*\top}g(x^*) + \nabla^2\left(y^{*\top}g(x^*)p^*\right) = 0.$$

Because of Assumption (A3), this gives

$$\theta = \frac{\alpha}{r^*}. \tag{25}$$

If $\alpha = 0$, then $\theta = 0$, and so from (13) and (14) and $\beta = 0$, it follows that $0 = \eta = \xi$. Hence, $(\alpha, \beta, \theta, \xi, \eta) = 0$ which contradicts (20). Hence, $\alpha > 0$ and from (25), $\theta > 0$. Using $\theta > 0$, $\alpha > 0$, and $\beta = 0$, (21) yields

$$p^* = 0.$$

This gives

$$f(x^*) = f(x^*) - \frac{1}{2} p^{*\top} \nabla^2 f(x^*) p^*.$$

Using $\theta > 0$, $\beta = 0$, and $p^* = 0$, (13) gives

$$g(x^*) \leq 0.$$

Thus, x^* is feasible for (NP), and the objective functions of (NP) and (ND) are equal.

If, for all (x, u, y, p), $f(.)$ is pseudobonvex and $y^{*\top} g(.)$ is semi-strictly pseudobonvex, by Theorem 2.1, x^* is an optimal solution of (NP).

3. SPECIAL CASE

If $p = 0$, the dual problem (ND) reduces to the following dual problem, considered by Weir and Mond [6]:

$$\begin{aligned} \text{maximize} \quad & f(u) \\ \text{subject to} \quad & r\nabla f(u) + \nabla \left(y^\top g(u) \right) = 0, \\ & y^\top g(u) \geq 0, \quad (r, y) \geq 0, \quad (r, y) \neq 0. \end{aligned}$$

REFERENCES

1. O.L. Mangasarian, Second and higher order duality in nonlinear programming, *J. Math. Anal. Appl.* **51**, 607–620 (1975).
2. O.L. Mangasarian, *Nonlinear Programming*, McGraw Hill, New York, (1969).
3. B. Mond, Second-order duality for nonlinear programs, *Opsearch* **11**, 90–99 (1974).
4. B. Mond and T. Weir, Generalized convexity and higher order duality, *Journal of Mathematical Sciences*, 16–18, 74–92 (1981–83).
5. C.R. Bector and S. Chandra, Generalized-bonvex function and second-order duality in mathematical programming, Research Report, 83-2, Dept. of Actuarial and Management Sciences, The University of Manitoba, Winnipeg, (January 1995).
6. T. Weir and B. Mond, Sufficient Fritz John optimality conditions and duality for nonlinear programming problems, **23** (3), 129–141 (1986).